Supplement to "Accuracy of the Tracy-Widom limits for the extreme eigenvalues in white White matrices"

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Abstract

In this note, we provide proofs to Lemma 1, Lemma 3, and Eq.(72) of the paper. All the notation follows the paper.

1 Proof of Lemma 1

In the proof, we focus on $s^k G(s)$ and $s^k G'(s)$. Results for quantities related to G_N follow, for $\Delta_N = \mathrm{O}(1)$ and $G''(s) = sG(s)$.

Recall that $G = \frac{1}{\sqrt{2}}\mathrm{Ai}$. So, Eq.(61) and Eq.(62) of the paper imply that, for all $s \geq 0$,

$$|s^k G(s)|, |s^k G'(s)| \leq C s^{k+1/4} e^{-\frac{2}{3}s^{3/2}} \leq C e^{-\beta s}. \tag{1}$$

If $s_0 < 0$, we replace the constant C on the rightmost sides with

$$C(s_0) = \max\{C, \max_{s\in[s_0,0]} |s^k G(s)|, \max_{s\in[s_0,0]} |s^k G'(s)|\},$$

then the bounds in (1) hold with C replaced by $C(s_0)$, uniformly for all $s \geq s_0$. By definition, $C(s_0)$ is continuous and non-increasing.

2 Proof of Lemma 3

Bound on $|R'_N(\xi)|$. For $R_N(\xi) = (\zeta'(\xi)/\zeta'_N)^{-1/2}$ with $\zeta'_N = \zeta'(\xi_+)$, we have

$$R'_N(\xi) = -\frac{1}{2}(\zeta'_N)^{1/2}\zeta'(\xi)^{-3/2}\zeta''(\xi). \tag{2}$$

Thus, to derive the bound for $|R'_N(\xi)|$, we study ζ'_N, $\zeta'(\xi)$ and $\zeta''(\xi)$ in turn.

1° Consider ζ'_N first. Let $m_\pm = m \pm \frac{1}{2}$. Simple calculus shows, as $N \to \infty$, if $n/N \to \gamma$, then

$$\zeta'_N = \frac{n_+^{1/6} N_+^{1/6} (n_+ + N_+)^{1/3}}{2^{1/3}\left(\sqrt{n_+} + \sqrt{N_+}\right)^{4/3}} \longrightarrow \frac{\gamma^{1/6}(1+\gamma)^{1/3}}{2^{1/3}\left(1+\sqrt{\gamma}\right)^{4/3}} = \zeta'_\infty. \tag{3}$$

2° Switch to $\zeta'(\xi)$. Let $\xi_\pm^\infty = \lim_{N\to\infty}\xi_\pm = 2\left(1 \pm \sqrt{\gamma}\right)^2/(1+\gamma)$.

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Assume first that $s \geq 0$. Then the LG transformation implies that $\frac{2}{3}\zeta^{3/2} = \int_{\xi_+}^{\xi} \sqrt{f(z)}dz$. Hence, we have

$$\zeta'(\xi) = \left[\frac{3}{2}\int_{\xi_+}^{\xi} \sqrt{f(z)}dz\right]^{-1/3} \sqrt{f(\xi)} = \left[\frac{3}{2}\int_{\xi_+}^{\xi} \sqrt{z-\xi_+} \cdot \frac{\sqrt{z-\xi_-}}{2z}\, dz\right]^{-1/3} \sqrt{f(\xi)}.$$

Note that $(\xi_+ - \xi_-)^{1/2}/(2\xi) \leq (z-\xi_-)^{1/2}/(2z) \leq (\xi-\xi_-)^{1/2}/(2\xi_+)$. We thus obtain lower and upper bounds for $\zeta'(\xi)$ as

$$\frac{\xi_+^{1/3}(\xi-\xi_-)^{1/3}}{2^{2/3}\xi} \leq \zeta'(\xi) \leq \frac{(\xi-\xi_-)^{1/2}}{2^{2/3}\xi^{2/3}(\xi_+-\xi_-)^{1/6}}, \tag{4}$$

which hold uniformly for $s \in I_{1,N}$. As $N \to \infty$, both bounds in (4) converge to the same limit $2^{-2/3}(\xi_+^\infty)^{-2/3}(\xi_+^\infty - \xi_-^\infty)^{1/3} = \zeta'_\infty$, uniformly for $s \in I_{1,N}$.

If $s_0 < 0$, we focus on $s \in [s_0, 0]$, for the above discussion is valid for all $s \in [0, s_1 N^{1/6})$. Now, the LG transformation implies that $\frac{2}{3}(-\zeta)^{3/2} = \int_{\xi}^{\xi_+} \sqrt{-f(z)}dz$. So

$$\zeta'(\xi) = \left[\frac{3}{2}\int_{\xi}^{\xi_+} \sqrt{-f(z)}dz\right]^{-1/3} \sqrt{-f(\xi)} = \left[\frac{3}{2}\int_{\xi}^{\xi_+} \sqrt{\xi_+ - z} \cdot \frac{\sqrt{z-\xi_-}}{2z}\, dz\right]^{-1/3} \sqrt{-f(\xi)}.$$

Here, we have $\sqrt{\xi-\xi_-}/(2\xi_+) \leq \sqrt{z-\xi_-}/(2z) \leq \sqrt{\xi_+-\xi_-}/(2\xi)$. Similar argument to the above then shows that uniformly for $s \in [s_0, 0]$, $\zeta'(\xi) \to \zeta'_\infty$ as $N \to \infty$.

Therefore, we conclude that, for $C_1 < 1 < C_2$,

$$C_1\zeta'_\infty \leq \zeta'(\xi) \leq C_2\zeta'_\infty, \tag{5}$$

uniformly for $N \geq N(s_0, \gamma)$ and $s \in I_{1,N}$.

3° Now consider $\zeta''(\xi)$. To start with, we first derive a convenient representation for it. By the definition of ζ, we have $(\zeta')^2 = f\zeta^{-1}$. Take derivative with respect to ξ on both sides, and then divide both sides by $2\zeta'$ to obtain

$$\zeta'' = \frac{f'\zeta - f\zeta'}{2\zeta'\zeta^2}.$$

Furthermore, we replace ζ by $f/(\zeta')^2$ to obtain

$$\zeta''(\xi) = \frac{f'(\xi)\zeta'(\xi) - \zeta'(\xi)^4}{2f(\xi)} = \zeta'(\xi) \cdot \frac{f'(\xi) - \zeta'(\xi)^3}{\xi - \xi_+} \cdot \frac{2\xi^2}{\xi - \xi_-}. \tag{6}$$

For the three multipliers on the rightmost side, we have already studied ζ'. In what follows, we focus on the other two terms.

First assume that $s_0 \geq 0$. We have

$$\frac{f'(\xi) - \zeta'(\xi)^3}{\xi - \xi_+} = \frac{\mathcal{I}(\xi)}{\xi - \xi_+} + \frac{1}{4\xi^2} - \frac{\xi - \xi_-}{2\xi^3}, \tag{7}$$

with $\mathcal{I}(\xi) = (\xi - \xi_-)/(4\xi^2) - \zeta'(\xi)^3$. By (4), we have

$$\left[1 - \left(\frac{\xi-\xi_-}{\xi_+-\xi_-}\right)^{1/2}\right]\frac{\xi-\xi_-}{4\xi^2} \leq \mathcal{I}(\xi) \leq \left(1 - \frac{\xi_+}{\xi}\right)\frac{\xi-\xi_-}{4\xi^2}.$$

So, when $N \geq N_0(s_0, \gamma)$, for all $s \in I_{1,N}$, $|\mathcal{I}(\xi)/(\xi - \xi_+)| \leq (\xi - \xi_-)/\xi^3$. And hence,

$$\left|\frac{f'(\xi) - \zeta'(\xi)^3}{\xi - \xi_+}\right| \leq \frac{1}{4\xi^2} + \frac{2(\xi-\xi_-)}{\xi^3} \leq \frac{9}{4\xi^2} \leq \frac{C}{(\xi_+^\infty)^2}.$$

In addition, when $N \geq N_0(s_0, \gamma)$,

$$|\zeta'(\xi)| \leq \left(\xi_+^\infty - \xi_-^\infty\right)^{1/3} (\xi_+^\infty)^{-2/3}, \qquad \left|\frac{2\xi^2}{\xi_+ - \xi_-}\right| \leq \frac{4(\xi_+^\infty)^2}{\xi_+^\infty - \xi_-^\infty}.$$

The last three bounds together imply that

$$|\zeta''(\xi)| \leq C(\xi_+^\infty)^{-2/3}(\xi_+^\infty - \xi_-^\infty)^{-2/3} = C\gamma^{-1/3}(1+\sqrt{\gamma})^{-4/3}(1+\gamma)^{4/3}, \tag{8}$$

uniformly for $N \geq N_0(s_0, \gamma)$ and $s \in I_{1,N}$.

If $s_0 < 0$, we just focus on $s \in [s_0, 0]$. Using (7), by similar argument as in the study of $\zeta'(\xi)$, we obtain that (8) also holds for this case.

Combining the three parts, we obtain that

$$|R_N'(\xi)| \leq C\gamma^{-1/2}(1+\gamma), \tag{9}$$

uniformly for $N \geq N_0(s_0, \gamma)$ and $s \in I_{1,N}$.

Bound on $|R_N(\xi) - 1|$. By definition, we have $\xi = \xi_+ + s\tilde{\sigma}_{n,N}/\kappa_N$, with

$$\tilde{\sigma}_{n,N}/\kappa_N \leq 4(1+\gamma^{-1/2})^{1/3}(1+\gamma^{1/2})(1+\gamma)^{-1}N^{-2/3} = \mathrm{O}(N^{-2/3}).$$

By a first order Taylor expansion, we obtain

$$R_N(\xi) = R_N(\xi_+) + R_N'(\xi^*)(\xi - \xi_+),$$

for some ξ^* lying between ξ and ξ_+. As $R_N(\xi_+) = 1$, this implies $R_N(\xi) - 1 = R_N'(\xi^*)(\xi - \xi_+)$. And so

$$\begin{aligned} |R_N(\xi) - 1| &\leq |R_N'(\xi^*)| 4\left(1 + 1/\sqrt{\gamma}\right)^{1/3}\left(1+\sqrt{\gamma}\right)(1+\gamma)^{-1}N^{-2/3}|s| \\ &\leq C\gamma^{1/2}(1+\gamma^{1/2})(1+\gamma^{-1/2})^{1/3}N^{-2/3}|s| \\ &\leq CN^{-2/3}|s|, \end{aligned}$$

uniformly for $N \geq N_0(s_0, \gamma)$ and $s \in I_{1,N}$. The second inequality comes from (9), and the last comes from the fact that $\gamma \geq 1$.

Bound on $|\kappa_N^{2/3}\zeta - s|$. Expanding $\kappa_N^{2/3}\zeta(\xi)$ at ξ_+ to the second order, we obtain

$$\kappa_N^{2/3}\zeta(\xi) = \kappa_N^{2/3}\zeta(\tilde{\mu}_{n,N}/\kappa_N + s\tilde{\sigma}_{n,N}/\kappa_N) = \kappa_N^{2/3}\zeta(\xi_+) + \kappa_N^{-1/3}\tilde{\sigma}_{n,N}\zeta_N' s + \tfrac{1}{2}\kappa_N^{-4/3}\tilde{\sigma}_{n,N}^2\zeta''(\xi^*)s^2.$$

Recall that $\zeta(\xi_+) = 0$ and that $\tilde{\sigma}_{n,N}\kappa_N^{-1/3}\zeta_N' = 1$. We thus have

$$\kappa_N^{2/3}\zeta(\xi) - s = \frac{1}{2}\frac{\tilde{\sigma}_{n,N}}{\kappa_N}\frac{\zeta''(\xi^*)}{\zeta_N'}s^2.$$

By previous discussion, we have $\tilde{\sigma}_{n,N}/\kappa_N = \mathrm{O}(N^{-2/3})$ and $\zeta''(\xi^*)/\zeta_N' = \mathrm{O}(1)$, uniformly for $s \in I_{1,N}$. Thus, we obtain

$$|\kappa_N^{2/3}\zeta - s| \leq CN^{-2/3}s^2,$$

uniformly for $N \geq N_0(s_0, \gamma)$ and $s \in I_{1,N}$. Note that on $I_{1,N}$, $|s| \leq s_1 N^{1/6}$, and hence we could modify the above bound to

$$|\kappa_N^{2/3}\zeta - s| \leq (CN^{-2/3}s^2) \wedge \tfrac{1}{2}|s| \wedge 1.$$

This completes the proof of Lemma 3.

3 Proof of (72)

Bound on $\tilde{\sigma}_{n,N}\kappa_N^{-1}|R_N'/R_N|(\xi)$. Focus on $|R_N'/R_N|(\xi)$. By definition, we have

$$(R_N'/R_N)(\xi) = -\frac{1}{2}(\zeta''/\zeta')(\xi) = -\frac{1}{4}(f'/f)(\xi) + \frac{1}{6}\sqrt{f(\xi)}/I(\sqrt{f}),$$

where $I(\sqrt{f}) = \int_{\xi_+}^{\xi}\sqrt{f}$. For f'/f, we have for all $s \in I_{2,N}$,

$$\left|\frac{f'(\xi)}{f(\xi)}\right| = \left|\frac{1}{\xi-\xi_+} + \frac{1}{\xi-\xi_-} - \frac{2}{\xi}\right| \le \frac{4}{\xi-\xi_+} = \frac{4\kappa_N}{s\tilde{\sigma}_{n,N}} \le C\frac{\kappa_N}{\tilde{\sigma}_{n,N}}.$$

On the other hand, we have

$$\begin{aligned}\frac{\sqrt{f(\xi)}}{I(\sqrt{f})} &= \frac{(\xi-\xi_+)^{1/2}}{\int_{\xi_+}^{\xi}(z-\xi_+)^{1/2}\frac{\xi(z-\xi_-)^{1/2}}{z(\xi-\xi_-)^{1/2}}dz} \\ &\le \frac{(\xi-\xi_+)^{1/2}}{\int_{\xi_+}^{\xi}(z-\xi_+)^{1/2}dz} = \frac{3}{2(\xi-\xi_+)} \le \frac{3}{2}\frac{\kappa_N}{\tilde{\sigma}_{n,N}}s_1^{-1}N^{-1/6} \le C\frac{\kappa_N}{\tilde{\sigma}_{n,N}}.\end{aligned}$$

Then, the triangle inequality shows that $|R_N'/R_N|(\xi)$ is bounded by $C\kappa_N/\tilde{\sigma}_{n,N}$, and so

$$r_N\tilde{\sigma}_{n,N}\kappa_N^{-1}|R_N'/R_N|(\xi) \le C,$$

uniformly for $N \ge N_0(\gamma)$, and $s \in I_{2,N}$.

Bound on $R_N(\xi)\mathcal{M}(\kappa_N^{2/3}\zeta)$. For this term, we first introduce the following lemma.

Lemma S.1. *Let $r > 0$ be fixed, $x = x_N(s) = \tilde{\mu}_{n,N} + s\tilde{\sigma}_{n,N}$, and $\xi = x/\kappa_N$. Then for $s \ge r^2$, we have*

$$\tilde{\sigma}_{n,N}\sqrt{f(\xi)} \ge r\xi_+/\xi = r\tilde{\mu}_{n,N}/(\tilde{\mu}_{n,N} + s\tilde{\sigma}_{n,N}).$$

Proof. Observe that $(\xi_+ - \xi_-)/(4\xi_+^2) = \kappa_N/\tilde{\sigma}_{n,N}^3$. So, when $s \ge r^2$,

$$\sqrt{f(\xi)} = \frac{(\xi-\xi_+)^{1/2}(\xi-\xi_-)^{1/2}}{2\xi} \ge r\frac{\tilde{\sigma}_{n,N}^{1/2}}{\kappa_N^{1/2}}\frac{(\xi_+-\xi_-)^{1/2}}{2\xi_+}\frac{\xi_+}{\xi} = \frac{r\xi_+}{\tilde{\sigma}_{n,N}\xi}.$$

We complete the proof by multiplying both sides with $\tilde{\sigma}_{n,N}$. □

Note that $R_N(\xi) = \kappa_N^{1/6}\tilde{\sigma}_{n,N}^{-1/2}\hat{f}^{-1/4}(\xi)$, and that Eq.(62) of the paper implies that $|\mathcal{M}(\kappa_N^{2/3}\zeta)| \le C\kappa_N^{-1/6}\zeta^{-1/4}$ uniformly on $I_{2,N}$. Thus, as $s_1 \ge 1$, we apply Lemma S.1 with $r = 1$ to obtain that

$$\begin{aligned}R_N(\xi)\mathcal{M}(\kappa_N^{2/3}\zeta) &\le C\zeta^{-1/4}\tilde{\sigma}_{n,N}^{-1/2}\hat{f}^{-1/4}(\xi) = Cf^{-1/4}(\xi)\tilde{\sigma}_{n,N}^{-1/2} \\ &\le C\left(\frac{\tilde{\mu}_{n,N}}{\tilde{\mu}_{n,N}+s\tilde{\sigma}_{n,N}}\right)^{-1/2} \le Cs,\end{aligned}$$

uniformly for $s \in I_{2,N}$.